Groups acting on themselves by left multiplication

Recall: A group G has a natural action on itself by left multiplication.

i.e. $g \cdot h := gh$.

(We'll soon see that this action is both faithful and transitive.)

Ex: Let $G = \{1, a, b, c\}$ be the non-cyclic group of order 4, iso. to $\mathbb{Z}_2 \times \mathbb{Z}_2$, also called the Klein 4-group.

Trick: label the elements g_1, g_2, g_3, g_4, respectively so that we can think of each element as a permutation of $\{1, 2, 3, 4\}$.

i.e. $\sigma_g(i) = j \Leftrightarrow g g_i = g_j$.

Then
$$g_2 \cdot g_1 = g_2 \quad \text{so} \quad \sigma_{g_2}(1) = 2$$
$$g_2 \cdot g_2 = g_1 \quad \text{so} \quad \sigma_{g_2}(2) = 1$$
$$g_2 \cdot g_3 = g_4 \quad \text{so} \quad \sigma_{g_2}(3) = 4$$
$$g_2 \cdot g_4 = g_3 \quad \text{so} \quad \sigma_{g_2}(4) = 3$$

Thus, $\sigma_{g_2} = (1\,2)(3\,4)$

In the permutation representation of this action, we similarly compute that $g_1 = 1 \longmapsto \sigma_{g_1} = 1$, $g_2 \longmapsto \sigma_{g_2} = (1\,2)(3\,4)$, $g_3 \longmapsto \sigma_{g_3} = (1\,3)(2\,4)$, $g_4 \longmapsto \sigma_{g_4} = (1\,4)(2\,3)$

More generally, if $H \leq G$, G acts on the set A of left cosets of H by left multiplication! i.e. $g \cdot hH = ghH$.

We can check this is actually a group action:

$$1 \cdot hH = 1hH = hH \quad \text{and} \quad \text{if } g, g' \in G, \ (gg')\cdot hH = (gg')hH = g(g'hH)$$
$$= g\cdot(g'\cdot hH) \ \checkmark$$

Note: This will not in general be a faithful action:

Ex: If G is abelian, and $h \in H$, then for any coset gH,

$h \cdot gH = g\underbrace{hH}_{= H} = gH$. i.e. H is in the kernel of the action.

Ex: $G = D_8$, $H = \langle s \rangle$.

There are 4 distinct cosets: $1H, rH, r^2H, r^3H$. Label them 1, 2, 3, 4.

Then $s \cdot 1H = sH = 1H \Rightarrow \sigma_s(1) = 1$

$s \cdot rH = srH = r^3H \Rightarrow \sigma_s(2) = 4$

$s \cdot r^2H = sr^2H = r^2H \Rightarrow \sigma_s(3) = 3$

$s \cdot r^3 = sr^3H = rH \Rightarrow \sigma_s(4) = 2$

Thus $\sigma_s = (2\ 4)$

Similarly, we compute that $\sigma_r = (1\ 2\ 3\ 4)$.

Since the corresponding permutation representation is a homomorphism

$D_8 \longrightarrow S_n$, we can determine where the rest of D_8 goes, since we know where its generators go.

(i.e. $\sigma_{s^i r^j} = \sigma_s^{\,i} \sigma_r^{\,j}$)

Since we know a lot about the structure of a group, we in turn know a lot about this action:

Thm: G a group, $H \leq G$, and let G act on the set A of left cosets of H by left multiplication. Let $\pi_H : G \longrightarrow S_A$ be the associated permutation representation. Then

1.) G acts transitively on A,

2.) The stabilizer in G of $1H \in A$ is H,

3.) the kernel of the action (i.e. $\ker \pi_H$) is $\bigcap_{x \in G} xHx^{-1}$ and $\ker \pi_H$ is the largest normal subgroup of G contained in H.

Pf:

1.) Let $gH, hH \in A$. Then $(hg^{-1}) \cdot gH = hg^{-1}gH = hH$, so G acts transitively.

2.) $g \cdot 1H = 1H \iff gH = 1H \iff g \in H$.

3.)
$$\begin{aligned}
\ker \pi_H &= \{ g \in G \mid g \cdot xH = xH \ \forall \ x \in G \} \\
&= \{ g \in G \mid gxH = xH \ \forall \ x \in G \} \\
&= \{ g \in G \mid gx \in xH \ \forall \ x \in G \} \\
&= \{ g \in G \mid g \in xHx^{-1} \ \forall \ x \in G \} \\
&= \bigcap_{x \in G} xHx^{-1}
\end{aligned}$$

To show that $\ker \pi_H$ is the largest normal subgroup of G in H, We first note that it's normal since it's the kernel of a map, and $\ker \pi_H \leq H$ since if $g \cdot H = H$ then $g \in H$.

If $N \trianglelefteq G$ and $N \leq H$, then $\forall x \in G$, $N = xNx^{-1} \leq xHx^{-1}$

$\Rightarrow N \leq \bigcap xHx^{-1} = \ker \pi_H$. $\square$

Now we can prove a big result in group theory:

<u>Cayley's Theorem</u>: Every group is isomorphic to some subgroup of a symmetric group. If $|G| = n < \infty$, then G is isomorphic to a subgroup of S_n.

<u>Pf</u>: Set $H = 1$ and apply the previous theorem. Then

$$\pi_H : G \to S_A \text{ , where } A = \text{the cosets of } 1 = G.$$

$\ker \pi_H = \bigcap x 1 x^{-1} = 1 \Rightarrow \pi_H$ is injective. $\square$

Historically, finite groups were only studied as subgroups of S_n!

We can also now prove another result that will help us classify finite groups:

<u>Cor</u>: If G is a finite group of order n, and p is the smallest prime dividing n, then any subgroup of index p is normal.
(Note: we already showed this in the case $p = 2$.)

Pf: Suppose $H \leq G$ and $|G:H| = p$. Let G act on left cosets of H, and π_H the corresponding permutation representation.

Let $K = \ker \pi_H$. Then $K \leq H$. Let $|H:K| = k$. Then

$|G:K| = |G:H||H:K| = pk$ (by applying Lagrange's Thm three times)

Since H has p left cosets, $\pi_H : G \to S_p$, so G/K is isomorphic to $\pi_H(G) \leq S_p$. Thus $|G/K| \mid |S_p| \implies pk \mid p! \implies k \mid (p-1)!$

But $k \mid |G|$ so every prime dividing k must be $\geq p$.
But the only primes dividing $(p-1)!$ are $< p$. Thus, $k = 1$.

$\implies |H:K| = 1 \implies H = K$, which is normal! $\square$